ISRAEL JOURNAL OF MATHEMATICS 76 (1991), 299–316

TOPOLOGICAL AND METRIC PROPERTIES OF INFINITE CLUSTERS IN STATIONARY TWO-DIMENSIONAL SITE PERCOLATION

BY

R. M. BURTON*

Department of Mathematics
Oregon State University, Corvallis, OR 97331, USA

AND

M. KEANE**

Department of Mathematics and Informatics
Delft University of Technology
Mekelweg 4, 2628 CD Delft, The Netherlands

ABSTRACT

A dependent percolation model is a random coloring of the two-lattice. It is assumed that there are a finite number of colors and that the coloring is translation invariant. Each color defines a random subset of the lattice. Connected components of this subset are called clusters. This paper gives a classification of the infinite behavior of these clusters. In particular, it is shown that the lattice is divided into disjoint infinite strips, lying adjacently. Each strip is either composed of an infinite cluster together with isolated finite clusters or else is entirely composed of finite clusters. Examples of the various types of behavior are constructed.

* Research supported in part by AFOSR Grant 91-0215, the Alexander van Humboldt Foundation and the SFB-170, University of Göttingen.

** Research supported in part by the SFB-170, University of Göttingen.

Received September 11, 1989

Introduction

In two-dimensional site percolation, the basic object of interest is a random coloring of the points of $\mathbb{Z}^2$, provided by a probability measure on the set of all possible colorings of the points of $\mathbb{Z}^2$ using two colors. The group structure of $\mathbb{Z}^2$ enters by the requirement that the underlying probability measure be *stationary*, i.e. invariant under translations by elements of this group; the intuitive meaning of this assumption is that we may "forget" which point of the coloring represents the origin in $\mathbb{Z}^2$, preserving only the relative positions of points. Given such a coloring, say, with the colors black and white, $\mathbb{Z}^2$ falls apart into black clusters and white clusters, a *cluster* being a maximal connected unicolored subset of $\mathbb{Z}^2$ viewed as a nearest neighbor graph. One says that (black or white) percolation occurs if infinite (black or white) clusters occur with positive probability. If percolation occurs, it is natural to ask how many infinite clusters appear in such a random coloring, how the topological form of these clusters can be described, and how they fit together in $\mathbb{Z}^2$. A more quantitative question concerns their individual densities in $\mathbb{Z}^2$: Do these densities exist, and if so, are they equal to zero, or positive, or can both possibilities coexist? In this article we determine all possible behavior for two-dimensional site percolation and indicate for all cases except one how examples can be constructed for a given behavior. With no loss of generality, the underlying probability measure may be assumed to be ergodic, as any behavior which can be observed in the stationary case can also be observed in the ergodic case, by ergodic decomposition. Intuitively, the ergodicity assumption (which is purely for convenience) means that two random colorings drawn from the same underlying probability mechanism will have the same properties which are invariantly defined (i.e. which do not depend on where the origin lies). For example, the number of infinite clusters of a given color, or the maximal upper density of the infinite clusters, are quantities which take the same value for almost every coloring, and thus depend only on the underlying ergodic probability measure.

The article consists of three sections, whose contents are as follows. In the first section, we consider clustering for any fixed coloring of $\mathbb{Z}^2$ by two colors, denoted 0 and 1. An enclosure relation between clusters is defined, and a cluster is called essential if it is not enclosed by another cluster. We show that either all clusters are finite and unessential, in which case the coloring is called an infinite cascade, or every cluster is contained in a unique essential cluster, in which case the coloring is called essential. In the essential case, we distinguish three types of infinite objects:

- 0-ribbons: consisting of the union of all clusters enclosed by a fixed infinite essential cluster of color 0,
- 1-ribbons: consisting of the union of all clusters enclosed by a fixed infinite essential cluster of color 1,
- quilts (or q-ribbons): which are infinite maximal *-connected (= connected or diagonally connected) sets in the union of all clusters enclosed by all finite essential clusters,

and one type of finite object:

- rocks, which are the finite maximal *-connected sets in the preceding union.

Each coloring of $\mathbb{Z}^2$ thus induces a partition of $\mathbb{Z}^2$ into 0-ribbons, 1-ribbons, quilts, and rocks, which contains all necessary information for describing infinite clusters and their positions. This section contains essentially definitions only.

In section two, we assume that the coloring is random, according to an underlying stationary ergodic probability measure. Then we show that the following statements are true with probability one:

- Each rock has exactly two ribbons (one 0-ribbon and one 1-ribbon) as *-neighbors; thus rocks can be "eliminated" without disturbing the topological ribbon picture.
- Each ribbon has at most two ribbons as neighbors, one on each side; thus ribbons are topologically strips, half-planes or the full plane.
- Each ribbon and each cluster has a well-defined density.

In particular, the proof of the last statement is valid in $\mathbb{Z}^d$ for any dimension d; this answers a question of Newman and Shulman [6], and shows that rough clusters cannot exist in any stationary model in any dimension. In two dimensions, we also show that density zero and positive density cannot coexist; this is not true for larger dimensions. (For example, consider a model in $\mathbb{Z}^2$ with both finite clusters and an infinite cluster. Taking these as cross-sections of cylinder in $\mathbb{Z}^3$ gives us infinite clusters of both zero and positive density.) We conclude this section by distinguishing five different situations which can occur. The **order type** of an ergodic measure is defined to be $\{1, 2, \cdots N\}, \mathbb{N}$, or $\mathbb{Z}$, depending on the number N and ordering of ribbons occurring. If the order type is $\{1, \cdots, N\}$

or $\mathbb{N}$, then all infinite clusters and ribbons have positive density, and the way in which they neighbor each other is given by a fixed sequence $\xi = (\xi_1, \ldots, \xi_N)$ or $\xi = (\xi_0, \xi_1, \ldots)$ of symbols $0, 1$ and q, with $\xi_i \neq \xi_{i+1}$ for all i, which we call a **chart**. If the order type is $\mathbb{Z}$, then either all infinite clusters and ribbons have positive density, in which case again a fixed chart $\xi = (\cdots, \xi_{-1}\xi_0, \xi_1, \ldots)$ describes their neighbor structure, or all ribbons have zero density. In this case, the charts of points may differ. In all cases we know, the behavior can be described as an ergodic shift-invariant probability measure on $\{0, 1, q\}^{\mathbb{Z}}$, which we call a **representing measure.** We conjecture that every ergodic probability measure with order type $\mathbb{Z}$ and ribbons of zero density has a representing measure.

The third section is devoted to constructions. From work of Harris [4], as well as Fisher [3] and Toth [8], it follows that in the case of independent site percolation, with critical value $p_c > \frac{1}{2}$, there are four different behaviors. If $p < 1 - p_c$, there is a unique 0-ribbon; if $1 - p_c < p < p_c$, there is a unique quilt; if $p > p_c$, there is a unique 1-ribbon; and at p_c and $1 - p_c$ we have an infinite cascade, all with probability one. In Constructions 1 through 4 we show that all of the order types, charts and representing measures described above can occur in stationary ergodic models.

We would like to express our appreciation to the SFB-170 and to the Institut fur Mathematische Stochastik of the University of Göttingen for their support and hospitality. We also would like to thank A. Gandolfi for his comments.

1. Infinite Clustering in the Discrete Plane

This section is topological in nature. It contains a description of infinite clustering for possible realizations of two-dimensional site percolation.

We begin by recalling the basics of clustering in $\mathbb{Z}^2$. The **distance** between points of $\mathbb{Z}^2$ is their L^1-distance. Two points of $\mathbb{Z}^2$ are **neighbors** if they are at distance one from each other. Thus, the distance between two points is the minimum of the number of jumps required to go from one to the other, by always jumping to a neighbor. Such a sequence of jumps is commonly called a **path,** whose **length** is the number of jumps.

If S is a subset of $\mathbb{Z}^2$, then the saturation of the relation "z and z' are neighbors" on S is an equivalence relation, whose equivalence classes are called **components of** S. We distinguish **finite components,** i.e. those of finite cardinality, from **infinite components.** For our purposes, the following definitions of the **in-**

terior, exterior, and closure of S, will be convenient; they may differ from those used elsewhere. We set

$$\begin{aligned}\mathrm{int}(S) &:= \bigcup\{K : K \text{ finite component of } \mathbb{Z}^2 \setminus S\},\\ \mathrm{ext}(S) &:= \bigcup\{K : K \text{ infinite component of } \mathbb{Z}^2 \setminus S\},\\ \mathrm{cl}(S) &:= S \bigcup \mathrm{int}(S).\end{aligned}$$

Next, we consider a fixed realization of site percolation. That is, let x denote any mapping from $\mathbb{Z}^2$ to $\{0,1\}$. For $i \in \{0,1\}$ we define

$$G_i := \{z \in \mathbb{Z}^2 : x(z) = i\}.$$

The components of G_i are called i-**clusters**, and a **cluster** is either a 0-cluster or a 1-cluster. The collection of clusters partitions $\mathbb{Z}^2$. If C and C' are clusters and $C \subseteq \mathrm{cl}(C')$, then we write $\underline{C \prec C'}$ and say that C' **encloses** $\underline{C}$, or $\underline{C}$ **is enclosed by** $\underline{C'}$. The enclosure relation thus defined is a partial ordering on the set of clusters; maximal elements of this ordering are called **essential clusters**, and the other clusters are **inessential**. Note that each inessential cluster is finite.

Definition: The realization x is called

(i) **essential** if each cluster is enclosed by an (unique) essential cluster, and

(ii) an **infinite cascade** if each cluster is inessential [and finite].

Clearly, a realization cannot be both essential and an infinite cascade. ∎

PROPOSITION 1: *A realization is either essential or an infinite cascade.*

Proof: If x is not essential, then there exists an infinite chain

$$C_1 \prec C_2 \prec \cdots$$

of distinct clusters, which are necessarily finite. It suffices to show that any $z \in \mathbb{Z}^2$ belongs to $\mathrm{cl}(C_n)$ for some n. Now suppose that $z \in \mathrm{ext}(C_n)$ for each n, and let d_n be the distance from z to C_n. Since $C_n \subseteq \mathrm{int}(C_{n+1})$, any path from z to C_n must contain a shorter path from z to C_{n+1}, and thus $d_{n+1} \leq d_n - 1$, which is absurd. ∎

We now concentrate on the further structure of an essential realization x. Two points of $\mathbb{Z}^2$ are ***-neighbors** if their *Euclidean* distance is either 1 or $\sqrt{2}$. Set

$$H = \bigcup\{\mathrm{cl}(C) : C \text{ is an essential finite cluster}\}.$$

The saturation of the relation "z and z' are *-neighbors" on H is an equivalence relation on H. We call the finite equivalence classes of this relation **rocks**, and the infinite equivalence classes **quilts** or **q-ribbons**. The proof of the following proposition is easy and we omit it.

PROPOSITION 2: *If x is an essential realization, then $\mathbb{Z}^2$ is partitioned into 0-ribbons, 1-ribbons, q-ribbons (=quilts), and rocks, such that:*
(i) *Ribbons of the same type are never neighbors;*
(ii) *Quilts and rocks are never *-neighbors;*
(iii) *Ribbons are infinite;*
(iv) *Rocks are finite;*
(v) *The closure of any cluster is contained in a ribbon or a rock.*

It seems that a procedure similar to the above one should be possible for realizations of site percolation in $\mathbb{Z}^d, d \geq 3$.

2. Stationary clustering

In the preceding section, we have defined for essential x a partition of $\mathbb{Z}^2$ into rocks and ribbons, reflecting the topological nature of the infinite clusters of x. If we introduce now the stochastic nature of percolation by assuming that x has been obtained from a $\mathbb{Z}^2$-stationary stochastic process, then much more can be said about the way in which the elements of the partition fit together, and we can investigate the statistical properties of ribbons, such as densities. Thus, let

$$X := \{x : x \text{ maps } \mathbb{Z}^2 \text{ to } \{0,1\}\},$$

and let μ be a probability measure on X which is stationary and ergodic under the action of $\mathbb{Z}^2$ on X by translation. Since we are investigating the possible types of behavior, the ergodicity assumption should be seen as one made purely for convenience of exposition, as any stationary measure may be decomposed into its ergodic components. Clearly, the event that $x \in X$ is an infinite cascade is $\mathbb{Z}^2$-invariant, and hence has probability 0 or 1 by ergodicity. In the section on examples we shall produce some μ for which almost all points are infinite cascades. Here we concentrate on the case in which x is essential μ-almost surely, i.e. μ is **essential.**

Our first result shows that rocks "can be eliminated", i.e. play no part in differentiating the structure of ribbons. Let R denote the event that there exists a rock in x with three or more ribbons as *-neighbors. (A rock cannot have a q-ribbon as a *-neighbor.)

THEOREM 1: $\mu(R) = 0$.

Proof: By the ergodic theorem, if $\mu(R)$ were positive (and hence equal to one), then there would exist an $\varepsilon > 0$ such that for μ-almost every $x \in X$, sufficiently large boxes $\Lambda_n = [0,n] \times [0,n] \subseteq \mathbf{Z}^2$ would contain at least εn^2 rocks with at least three ribbons as *-neighbors. This is now seen to be impossible, as follows. For each such rock, choose three ribbons. Since each of these ribbons is infinite, they each contain a path from a *-neighbor of the rock to the boundary of Λ_n; these paths end in three different boundary points. If we now start from $(0,0)$ on the boundary of Λ_n and traverse the boundary in a clockwise manner, then we call the second of these three points we meet the **central** points. An elementary application of the Jordan curve theorem now shows that the central points of different rocks must be distinct. Hence the boundary of Λ_n contains at least εn^2 points, which is absurd for large n.

It follows that each rock has exactly one 0-ribbon and one 1-ribbon as its *-neighbors, and hence plays no role in the topological structure of the ribbons. One way to formulate this is as follows. We determine a mapping $\varphi : X \to X, \varphi(x) = \tilde{x}$, by

$$\tilde{x}(z) := \begin{cases} 0 & \text{if } z \text{ belongs to a 0-ribbon,} \\ 1 & \text{if } z \text{ belongs to a 1-ribbon,} \\ x(z) & \text{if } z \text{ belongs to a quilt,} \\ 0 & \text{if } z \text{ belongs to a rock.} \end{cases}$$

Let $\tilde{\mu}$ be the image of μ under φ.

COROLLARY: *$\tilde{\mu}$ is $\mathbf{Z}^2$-stationary and ergodic. Moreover, with $\tilde{\mu}$-probability one, $\tilde{x}$ contains no rocks. Events described in terms of the infinite topology of ribbons have the same μ- and $\tilde{\mu}$-probabilities.*

The proof is omitted. We now show that ribbons are topologically infinite strips, half-planes, or full planes, with exactly one neighbor on each side. Let Q denote the event that x contains a ribbon whose complement in $\mathbf{Z}^2$ has at least three components.

THEOREM 2: $\mu(Q) = 0$.

Proof: The idea is the same as in the proof of Theorem 1, but more care is needed. Suppose that $\mu(Q)$ is positive (and hence equal to one). Then there is a ribbon type, say 0-ribbon, an integer N, and a positive number α such that the μ-probability that the origin is contained in a 0-ribbon and that there are three paths of length at most N from the origin through its 0-ribbon to three different components of its complement, is equal to α. It is easy to see that these three paths may be assumed to be disjoint, by increasing N and lowering α if necessary. (This is not essential to the proof, but it makes the Jordan curve theorem application easier.) Now choose K such that

$$\frac{K^2}{(K+2N)^2} > 1 - \frac{\alpha}{2},$$

and let M be much larger than K. In the box $\Lambda_M = [0, M] \times [0, M]$, insert disjoint squares of side length K at distance $2N$ from each other and from the boundary of Λ_M; we call these squares **K-squares**. Then there is an $\varepsilon > 0$, not depending on M, such that μ-almost surely for sufficiently large M at least εM^2 of the K-squares contain a point in an 0-ribbon with three disjoint paths to its complement as above. For each of these K-squares, pick such a point and three such paths, and continue the paths to the boundary of Λ_M through their corresponding complements in order to select three points of the boundary of Λ_M. As in Theorem 1, an application of the Jordan curve theorem shows that between any two central points belonging to different K-squares with the same 0-ribbons, there is a point on the boundary of Λ_M which lies in this 0-ribbon. Therefore the number of K-squares whose selected point lies in a given 0-ribbon is at most equal to the number of points on the boundary of Λ_M which belong to this 0-ribbon. Summing over all 0-ribbons yields $\varepsilon M^2 \leq 4M$, which is impossible for large M. A similar argument is valid for 1-ribbons and quilts. ∎

The meaning of $\mu(Q) = 0$ is that for μ-almost every point x of X, each ribbon in x is topologically a "strip", i.e. has at most two "sides" and exactly one neighbor on each side. As an application of this result, we show that the finite energy condition of [6] implies uniqueness of infinite clusters in two dimensions.

Definition: see [6]. For $z \in \mathbb{Z}^2$ and $i \in \{0, 1\}$ define $\tau : X \to X$ by

$$\tau(x)(z') := \begin{cases} x(z') & \text{if } z' \neq z, \\ i & \text{if } z' = z. \end{cases}$$

The measure μ is said to have **finite energy** if for any event A with $\mu(A) > 0$, any $z \in \mathbb{Z}^2$, and any $i \in \{0,1\}$,

$$\mu(\tau(A)) > 0.$$

If a measure has finite energy, then almost all ergodic components also have finite energy. Finite energy is equivalent to the distribution at the origin being nontrivial a.s. when conditioned on all other variables. But the σ-algebra of all other variables includes the σ-algebra of all shift invariant events.

COROLLARY: *If μ has finite energy, then μ-almost every $x \in X$ contains at most two ribbons.*

Proof: If not, then with positive probability x contains a ribbon with two neighbors, and changing the configuration on a large finite box then produces either a ribbon with two different neighbors on one side or a ribbon whose complement has three components, with positive probability, which is impossible because of Theorem 2. ∎

Up to this point we have only investigated the topological nature of ribbons. Now we turn to more quantitative stochastic properties. Our first result is independent of the dimension, and solves a problem in [6].

Definition: A subset S of $\mathbb{Z}^2$ has **density** α if for each sequence $R_1 \subseteq R_2 \subseteq \cdots$ of rectangles in $\mathbb{Z}^2$ with $\bigcup R_n = \mathbb{Z}^2$,

$$\lim_{n\to\infty} \frac{\#(S \cap R_n)}{\#(R_n)} = \alpha,$$

where $\#(\cdot)$ denotes cardinality. A subset which does not have **density** [α for any α] is called **rough.** ∎

THEOREM 3: *Let μ be stationary on X. Then with μ-probability one, each cluster and ribbon of x has density.*

Proof: We prove the theorem for infinite 0-clusters. A similar argument applies for 1-clusters and ribbons. With no loss of generality, we may assume that μ is ergodic.

The number N of infinite 0-clusters in a point x of X is μ-almost surely constant, by ergodicity with $0 \le N \le \infty$. Fix $x \in X$ with N infinite 0-clusters. If

R is a rectangle in $\mathbf{Z}^n$, each infinite 0-cluster C has $m(C,R) = \#(C \cap R)$ points lying in R; let

$$m_1(R) \geq m_2(R) \geq \cdots$$

designate the numbers $m(C,R)$ in some non-increasing order with their proper multiplicities. The **rank** of C in R is the index corresponding to C. For any $K < N+1$, we set

$$F_R(x) := m_1(R) + \cdots + m_K(R).$$

Then F, seen as a stochastic process, with underlying probability space (X,μ), indexed by rectangles in $\mathbf{Z}^2$, is **subadditive**. By the multiparameter subadditive ergodic theorem ([1], see also [5]) there exists a constant $\gamma = \gamma(K) \geq 0$ such that for any sequence $R_1 \subseteq R_2 \subseteq \cdots$ of rectangles with $\bigcup R_n = \mathbf{Z}^2$,

$$\lim_{n\to\infty} \frac{1}{\#(R_n)} F_{R_n} = \gamma$$

μ-almost surely. Hence the "density" of the rank K cluster in R exists and is equal to $\gamma(K) - \gamma(K-1) =: \alpha(K)$ (where $\gamma(0) = 0$). Now suppose that R_n is such that

$$\frac{\#(R_{n+1} \setminus R_n)}{\#(R_n)} \to 0.$$

If C is a fixed infinite 0-cluster of x which does not have density zero, and if K_n denotes the rank of C in R_n, it now follows easily that

$$\alpha(K_n) = \alpha(K_{n+1}) > 0$$

for all sufficiently large n, which yields the existence (and value) of the density of C. ∎

COROLLARY: *In stationary site percolation in $\mathbf{Z}^d$, all clusters have density with probability one.*

Although the above proof is valid for general dimensions, the following result is two-dimensional, and counter-examples in dimensions greater than two are easily constructed with the techniques of section 3.

THEOREM 4: *Let μ be stationary and ergodic on X. Then either all clusters and ribbons have positive densities with probability one, or all have zero density with probability one.*

Proof: Suppose that ribbons with positive and zero densities coexist. Then for some $\varepsilon > 0$, the event that the origin is contained in a ribbon of zero density which is a neighbor of a ribbon of density at least ε, would have positive μ-measure. This contradicts the fact that the frequency of this event in μ-almost every $x \in X$ must be zero, since there are at most $[1/\varepsilon]$ clusters of density at least ε, and hence at most $2[1/\varepsilon]$ neighbors (by Theorem 2) of density zero. A similar argument is valid in the case of clusters. ∎

We conclude this section by classifying the possibilities for neighboring ribbon sequences for essential μ. Five essentially different situations are distinguished, and we show by construction in the following section that four of these can occur. We suspect that the fifth situation cannot occur.

1. FINITE ORDER TYPE. Let μ be essential and ergodic, and let N denote the number of ribbons of μ. If N is finite, then μ has **finite order type** $I = \{1, 2, \cdots, N\}$. For μ-almost every $x \in X$, each of the N ribbons of x is a 0−, 1−, or q-ribbon, and the neighbor relation of Theorem 2 orders them to yield two (possibly equal) sequences

$$\xi = (\xi_1, \ldots, \xi_N) \in \{0, 1, q\}^N$$

and

$$\xi' = (\xi_N, \cdots, \xi_1) \in \{0, 1, q\}^N$$

such that $\xi_i \neq \xi_{i+1}$ for all $1 \leq i < N$. We call these sequences **charts**, and the lexicographically smaller one the **canonical chart** for μ. A chart is **admissible** if $\xi_i \neq \xi_{i+1}$ for all i. By ergodicity, the canonical chart is the same for μ-almost every $x \in X$. By Theorem 3, the corresponding ribbons and infinite clusters have positive densities. In Construction 2 of section 3 we show that all N and all ξ with $\xi_i \neq \xi_{i+1}$ can occur. It is interesting to note that if $\xi = \xi'$ (and N is odd), then it may happen either that the direction of a chart for $x \in X$ is determined (i.e. we know how to distinguish ribbons on one side of the central ribbon from those on the other side) or non-determined. If $\xi \neq \xi'$, then this direction is always determined.

2. ONE-SIDED INFINITE ORDER TYPE. Here $N = \infty$, but there is one ribbon with only one neighbor μ-almost surely. We say that μ has **order type** $I = \{0, 1, 2, \cdots\}$, and μ-almost every $x \in X$ possesses the same admissible **chart**

$$\xi = (\xi_0, \xi_1, \cdots) \in \{0, 1, q\}^N,$$

infinite with $\xi_i \neq \xi_{i+1}$ $(i \in \mathbb{N})$. Again all ribbons and infinite clusters have positive density with probability one (since the first one can be distinguished and hence has positive density). Construction 3 of section 3 shows that all possible charts can occur. In this situation, a direction for $x \in X$ is always determined.

3. TWO-SIDED INFINITE ORDER TYPE, POSITIVE DENSITY. In this situation, $N = \infty$ and every ribbon has two neighbors. The **order type** of μ is $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$, and all clusters and ribbons have positive density μ-almost surely. The possible **charts** for a point $x \in X$ consist of all translations and reflections around 0 of a single sequence $\xi = (\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots) \in \{0, 1, q\}^{\mathbf{Z}}$; to define a **canonical chart** for μ , we can put a cluster of maximal density at 0 such that no other cluster of maximal density lies to the left of 0, reducing the number of choices for ξ to two, and then proceed as in 1 with a different ordering to define ξ uniquely. If $\xi = \xi'$ then as in the finite order type case a direction may not be determined. Construction 3 of section 3 shows that all possibilities for ξ may occur.

4. TWO-SIDED INFINITE ORDER TYPE, ZERO DENSITY, REPRESENTING MEASURE. In this situation, $N = \infty$, every ribbon has zero density, and there is a measurable map

$$\varphi : X \to \{0, 1, q\}^{\mathbf{Z}}$$

such that for μ-almost all $x \in X, \varphi(x)$ is a **chart** for x (i.e. $\varphi(x)$ reflects the ordering of the ribbons of x up to translation and reflection), together with a shift-invariant ergodic probability measure ν on $\{0, 1, q\}^{\mathbf{Z}}$ such that $\varphi(\mu)$ and ν have the same sets of measure zero. We then say that ν represents μ. Such a ν is clearly unique. In Construction 4 of section 3, we show how to obtain an example for any given representing measure ν.

5. TWO-SIDED INFINITE ORDER TYPE, ZERO DENSITY, NO REPRESENTING MEASURE. We have not been able to show whether this case does or does not occur. We say $T\mu$ is **well-directed** if μ-almost every ribbon intersects the vertical axis in a finite set.

If μ is well-directed, then an explicit construction of a ν as in 4. is possible. We have no examples of μ with zero density ribbons that are not well-directed.

We close this section with the remark that by means of a simple modification, corresponding results for bond percolation models can be obtained.

3. Constructions

The central and most studied percolation model is independent percolation. Let $0 < p < 1$ and μ_p be product measure on X that assigns 1 with probability p and 0 with probability $1-p$ independently to each site in $\mathbb{Z}^2$. There is a value $p_c > 1/2$ (it is believed that $p_c = 0.59+$) called the **critical probability**. If $p > p_c$, then μ_p is a 1-ribbon; if $p_c > p > 1-p_c$, then μ_p is a quilt; and if $1-p_c > p$ then μ_p is a 0-ribbon. If $p = p_c$ or $1-p_c$, then μ_p is an infinite cascade such that if $x \in D_1 \prec D_2 \prec \cdots$ is a chain of unessential clusters, then for all large enough ν, D_ν are 1-clusters if $p = p_c$ (and 0-clusters if $p = 1-p_c$). These are due to the following factors which follow from arguments of Harris [4]. μ_p contains infinite *-connected 1 paths if and only if $p > 1-p_c$. μ_p contains arbitrarily large connected loops of 1's if and only if $p \geq p_c$. μ_p contains an infinite 1-cluster if and only if $p > p_c$. A complementary situation holds for 0-paths.

We conclude with a series of constructions which will provide proofs of the remaining claims from section 2.

Construction 1: Here we give a general construction that has very strong ergodic properties and allows one to make various examples. In particular we produce an example where a.s. every x belongs to an infinite cascade $x \in D_1 \prec D_2 \prec D_3 \prec \cdots$ of alternating 0-clusters and 1-clusters.

First we define a stationary ergodic $\tilde{\mu}$ on $\tilde{X} = \{\tilde{x} : \mathbb{Z}^2 \to \{0,1,2,\cdots\}\}$. To do this we define square configurations of points each of which have a label from $\{0,1,2,\ldots\}$. Let $l \geq 1, k \geq 2$ be integers.

A **1-stack** is an $L \times L$ configuration labeled 0.

A **1-frame** is an $h_1 \times h_1$ configuration where $h_1 = L+3$. The boundary points are labeled 1, there is a 1-stack as a subconfiguration and all other points are labeled 1. There are four different 1 frames.

Inductively, an n-**stack** is a $kh_{n-1} \times kh_{n-1}$ configuration made up of k^2 $(n-1)$-frames. An n-**frame** is an $h_n \times h_n$ configuration where $h_n = (kh_{n-1} + n + 2)$. The boundary points are labeled n, an n-stack occurs as a subconfiguration and all other points are labeled n. If there are N_{n-1} different $(n-1)$-frames, then there are $N_n = (n+1)^2(N_{n-1})^{k^2}$ different n-frames.

Note that if we have chosen an n-frame with equal probability from the N_n choices and if $k < n$ then each of the k-frames which occur as subconfigurations also occurs independently and with probability N_k^{-1}.

We describe the measure $\tilde{\mu}$ heuristically as a condition on the event that $x(0,0)$ belongs to an n-frame. Then the probability that $x(0,0)$ belongs to a given n-frame is N_n^{-1} which is independent of the choice of h_n^2 possible locations of $x(0,0)$ in the n-frame all of which also occur with equal probability. This n-frame occurs in an equally likely way as any of the k^2 possible locations in the $(n+1)$-stack and the choice of all other n-frames is independent of all choices made so far. This is enough to describe invariant probabilities on cylinder sets of $\tilde{X}$ and thus to define $\tilde{\mu}$. In one dimension similar constructions were first defined by Ornstein [7], while in $\mathbb{Z}^2$ by Burton [2]. It can be shown that $\tilde{\mu}$ has strong ergodic properties, e.g. tail triviality.

Let $\varphi : \{0,1,\cdots\} \to \{0,1\}$ by $\varphi(u) = 0$ if u is even and $\varphi(u) = 1$ if u is odd. This induces a transformation $\varphi : \tilde{X} \to X$ by $\psi(\tilde{x}) = x$ if $x(z) = \varphi(\tilde{x}(z))$. Let $\mu = \psi(\tilde{\mu})$ be the transported measure. Then μ almost all $x \in X$ have finite 1-clusters each with k^2 "holes" and thus is an infinite cascade. By construction μ is symmetric with respect to axis reflections and 90^0 rotations.

Construction 2: We construct an example with two infinite 0-clusters, one infinite 1-cluster and no finite clusters. That is we construct an example whose canonical chart is $(0,1,0)$. Then we will show how to modify this example so as to obtain models with any admissible chart ξ of order type $\{1,2,\ldots,n\}$.

Again we will describe a series of nested blocks. Each block will be a square configuration of points each labeled 0 or 1. Each block will have the property that considered as a subgraph of $\mathbb{Z}^2$ there is exactly one 1-cluster which will be a winding of width three and two 0-clusters one on each side of the 1-cluster. There will be six types of n-blocks for each $n = 0,1,2,\cdots$. 0-blocks will be 9×9 configurations of points. These are pictured in Fig. 1. The shaded region represents a strip of points of width three labeled 1, and the unshaded region represent points with 0-labels.

A type 1 1-block is a 15×15 configuration made up of 0-blocks of various types. The types of 0-blocks are arranged in a 5×5 pattern as given in Fig. 2. Figure 3 gives a schematic showing which points are labeled 1 (dark line) and which 0 (background). It is apparent that this 1-block is a 90^o clockwise rotation of a type 1 1-block. A type 3 1-block is constructed similarly. Its configuration is pictured in Fig. 4. A type k 1-block for $k = 4,5,6$ is a 90^o clockwise rotation of a type $k-1$ 1-block.

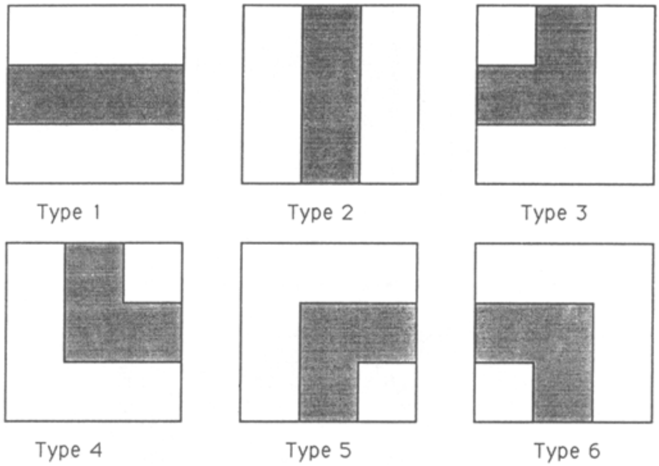

Fig. 1.

This construction is iterated. For example, a type 1 n-block is a $3 \cdot 5^n \times 3 \cdot 5^n$ configuration made up of 5^2 $(n-1)$-blocks whose types are arranged as in Figure 2.

5	6	5	1	6
2	2	2	5	3
3	2	2	2	5
5	3	2	2	2
4	1	1	4	3

Fig. 2.

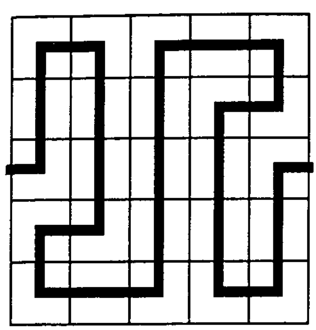

Fig. 3.

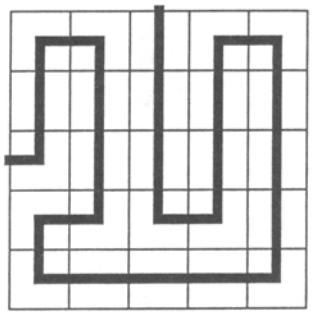

Fig. 4.

Let $P^{(n)} = [P_{ij}^{(n)}]$ be the 6×6 matrix such that $P_{ij}^{(n)}$ is the proportion of type j 0-blocks that occur in a type i n-block. $P^{(0)} = I$ and $P^{(n+m)} = P^{(n)}P^{(m)}$ so $P^{(n)} = [P^{(1)}]^n \cdot P^{(n)}$ is an irreducible stochastic matrix (in the rows); so $P^{(n)}$ converges to a stochastic matrix $Q = [Q_{ij}]$ such that Q_{ij} does not depend on i and Q_{ij} is the approximate proportion of j 0-blocks in any n block for large n. $P_{ij}^{(n)}$ is also the proportion of type j m-blocks contained in a type i $(n+m)$-block. This implies that as n goes to infinity the finite dimensional statistics of a type i n-block are the same for all i. Thus we can use these finite dimensional statistics to construct a measure μ on X which has $(0,1,0)$ as its chart.

Consider again that class of examples $(\tilde{\mu}, \tilde{X})$ in Construction 1. Take $\ell = 9, k = 5$. Because $\tilde{\mu}$-a.s. all $\tilde{x} \in \tilde{X}$ have 5×5 stacks of blocks in a nested construction, it is clear that we may relabel the $\tilde{x}(z)$ so as to embed Construction 2 into $\tilde{X}$. This gives a model with strong ergodic properties and chart $(0,1,0)$. We make a further modification to get the additional property of symmetry with respect to axis reflections and 90^o rotations. At each state of constructing an n-block we choose randomly one of the possible patterns of types of $(n-1)$-blocks that gives an n-block with two 0-clusters separated by a 1-cluster that touches the boundary at the correct places. At each stage we make all the choices independently and equiprobably.

Now let ξ be any admissible chart with order type $\{1, \ldots, n\}$. We make a similar construction as for order type $(0,1,0)$ but make the central strip of width $3n-6$ in 0-blocks of size $3n \times 3n$. Subdivide this strip into $n-2$ strips of width three. This gives a model that a.s. divides $\mathbb{Z}^2$ into n adjacent strips. Orient these strips. If $\xi_i = 0$, label all the points in strip i 0; if $\xi_i = 1$, label them 1; and if $\xi_i = Q$ label them in an alternating checkerboard pattern. This gives a μ with chart ξ.

Construction 3: To construct models of order type $\mathbb{N}$, modify the construction for finite order type. For concreteness we construct a model whose chart is $\xi = (0,1,0,1,\ldots)$. An n-block will be one of 6 types and will consists of 5^2 $(n-1)$-blocks as before but together with an extra outer layer. (For example, we do this when we put stacks into frames.) This outer layer will be at least 6 units thick and gives room to add an extra strip of points labeled 1. This is pictured in Fig. 5. Note that the central line wiggling through the diagram represents n distinct 1-clusters separated by 0-clusters while the upper line represents the new 1-cluster.

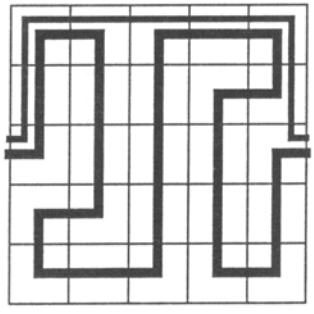

Fig. 5.

Models of order type $\mathbb{Z}$ whose clusters have positive density may be constructed by adding two new 1-clusters at each stage, one following the top boundary and one following the bottom boundary. This completes the classification described in section 2.

Construction 4: Finally we wish to indicate some zero density examples. Given ν as in situation 4 of section 2, define $\theta_y : \{0,1,q\}^{\mathbf{Z}} \to X$ by $\theta_y(\xi) = x$ where

$$x(m,n) = \begin{cases} 0 & \text{if} \quad \xi_{m'} = 0, \quad \text{if } \xi_{m'} = Q \quad \text{and } m+n \quad \text{is even} \\ & \qquad \text{where } m' = \text{ integer part of } (m+y)/3 \\ 1 & \text{otherwise.} \end{cases}$$

Let Y be a random variable taking an element of $\{0,1,2\}$ with equal probabilities, and let Y be independent of ν. Let μ be defined on X by setting $\mu = \nu^0\theta_y^{-1}$ conditioned on $Y = y$. This model has vertical strips which are 0-clusters, 1-clusters or checkerboard quilts. This model is symmetric with respect to reflection across the horizontal axis. It is symmetric with respect to reflection across the vertical axis if and only if ν and this time reversal have the same distribution. μ has rather poor ergodic properties, but by embedding it into $(\tilde{\mu}, \tilde{X})$ of Construction 1, these may be made as strong as desired.

References

[1] M. A. Akcoglu and U. Krengel, *Ergodic theorems for superadditive process,* J. Reine Angew. Math. **323** (1981), 53–67.

[2] R. M. Burton, *An asymptotic definition of K-groups of automorphisms and a non-Bernoullian counter example,* Z. Wahr. v. Geb. **47** (1979), 205–212.

[3] M. E. Fisher, *Critical probabilities for cluster size and percolation problems,* J. Math. Phys. **2** (1961), 620–627.

[4] T. E. Harris, *A lower bound for the critical probability in a certain percolation process,* Proc. Camb. Phil. Soc. **56** (1960), 13–20.

[5] U. Krengel, *Ergodic Theorems,* Walter de Gruyter, Berlin–New York, 1985, pp. 201–211.

[6] C. M. Newman and L. S. Schulman, *Infinite clusters in percolation models,* J. Stat. Phys. **26** (1981), 613–628.

[7] D. S. Ornstein, *A mixing transformation for which Pinsker's conjecture fails,* Adv. in Phys. **10** (1973), 103–123.

[8] B. Toth, *A lower bound for the critical probability of the square lattice site percolation,* Z. Wahr. v. Geb. **69** (1985), 19–22.